Amplitude and Phase: Second Order I

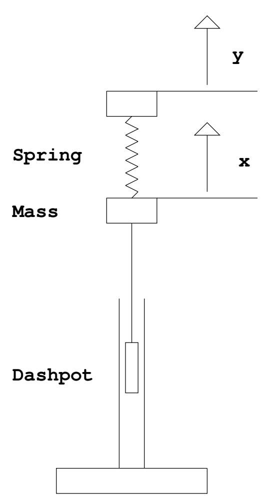
The model. The Mathlet Amplitude and Phase: Second order I illustrates a spring/mass/dashpot system that is driven through the spring. Suppose that y denotes the displacement of the plunger at the top of the spring, and $x(t)$ denotes the position of the mass, arranged so that $x = y$ when the spring is unstretched and uncompressed. There are two forces acting on the mass: the spring exerts a force force given by $k(y - x)$ (where k is the spring constant), and the dashpot exerts a force given by $-b\dot{x}$ (against the motion of the mass, with damping coefficient b). Newton's law gives

$$m\ddot{x} = k(y - x) - b\dot{x}$$

or, putting the system on the left and the driving term on the right,

$$m\ddot{x} + b\dot{x} + kx = ky\,. \qquad (1)$$

In this example it is natural to regard y, rather than ky, as the input signal, and the mass position x as the system response.



Sinusoidal input signals are fundamental. By starting the system at a peak when $t = 0$ we have

$$y = B\cos(\omega t)\,,$$

so the equation reads

$$m\ddot{x} + b\dot{x} + kx = kB\cos(\omega t)\,.$$

The Mathlet illustrates this system with $m = 1$ and $B = 1$, but we will carry out the analysis for general m and B.

The solution. Putting aside the possibility of resonance, we expect a sinusoidal solution, one of the form

$$x = A\cos(\omega t - \phi)$$

The ratio of the amplitude of the system response to that of the input signal, $g = A/B$, is called the *gain* of the system. We think of the system as fixed, while the frequency ω of the input signal can be varied, so the gain is a function of ω, $g(\omega)$. Similarly, the *phase lag* ϕ is a function of ω. The entire story of the steady state system response to sinusoidal input signals is encoded in those to functions of ω, the gain and the phase lag.

There is a systematic way to work out what g and ϕ are. The equation (1) is the real part of a complex-valued differential equation:

$$m\ddot{z} + b\dot{z} + kz = Bke^{st}$$

with $s = i\omega$. The Exponential Response Formula gives the solution

$$z_p = \frac{Bk}{p(s)}\, e^{st}$$

where

$$p(s) = ms^2 + bs + k$$

(as long as $p(s) \neq 0$).

Our choice of input signal and system response correspond in the complex equation to regarding Ae^{st} as the input signal and z_p as the exponential system response. The *transfer function* is the ratio between the two:

$$W(s) = \frac{k}{p(s)}$$

so

$$z_p = W(s)Ae^{st}\,.$$

Now take $s = i\omega$. The *complex gain* is

$$W(i\omega) = \frac{k}{k - m\omega^2 + ib\omega}\,. \tag{2}$$

I claim that the polar form of the complex gain determines the gain g and the phase lag ϕ as follows:

$$W(i\omega) = ge^{-i\phi}$$

To verify this, substitute this expression into the formula for z_p—

$$z_p = g\, e^{-i\phi}Be^{i\omega t} = gAe^{i(\omega t - \phi)}$$

—and extract the real part, to get the sinusoidal solution to (1):

$$y_p = gB\cos(\omega t - \phi)\,.$$

The amplitude of the input signal, B, has been multiplied by the gain

$$g(\omega) = |W(i\omega)| = \frac{k}{\sqrt{k^2 + (b^2 - 2mk)\omega^2 + m^2\omega^4}} \tag{3}$$

The phase lag of the system response, relative to the input signal, is $\phi = -\mathrm{Arg}(W(i\omega))$. Since $\mathrm{Arg}(1/z) = -\mathrm{Arg}(z)$, ϕ is the argument of the denominator in (2). The tangent of the argument of a complex number is the ratio of the imaginary part by the real part, so

$$\tan\phi = \frac{b\omega}{k - m\omega^2}$$

It's not quite correct to write $\phi = \arctan\left(\dfrac{b\omega}{k - m\omega^2}\right)$, since the arctan is chosen as the angle between $-\pi/2$ and $+\pi/2$ with given tangent, while the phase lag varies between 0 and π.

Questions. 1. It appears from the Mathlet that often, but not always, there is a nonzero frequency for which the gain is maximal. This is the "resonant frequency" ω_r. Compute this frequency, as a function of the system parameters. Explain why for some values of the parameters the only local maximum of gain is at $\omega = 0$.

2. At what frequency is the phase lag exactly 90°?

3. For all values of b and k, as ω gets large it appears that $A \to 0$. Is this right? Can you be more precise about how? That is, for ω very large, can you approximate A by a simpler expression that what we derived above?—maybe just a (negative) power of ω?

4. For all values of b and k, as ω gets large it appears that $\phi \to \pi$. Is this right?

5. How about the behavior of g and ϕ for small values of ω? Clearly $g(0) = 1$ and $\phi(0) = 0$, so linear approximation gives

$$g(\omega) \simeq 1 + a\omega \quad , \quad \phi(\omega) \simeq b\omega$$

for small ω. The Mathlet gives some indication of the values of a and b. What are they in fact? Does the Mathlet bear out your calculation?