Amplitude and Phase: First Order

Problem 1

(a) Express the real part of the complex valued function $\dfrac{e^{it/2}}{1+i}$ in the form $a\cos(\omega t) + b\sin(\omega t)$ and as $A\cos(\omega t - \phi)$ (where a, b, A, and ϕ are real). How is the pair a, b, related to the pair A, ϕ? Sketch graphs of $\cos(\omega t)$, $\sin(\omega t)$, and then $a\cos(\omega t) + b\sin(\omega t)$, and reconcile it with a graph of $A\cos(\omega t - \phi)$. (The Mathlet `Trigonometric Id` might be helpful to you here.)

(b) Around here, the ocean experiences tides. About twice a day the ocean level rises and falls by several feet. This is why small boats are often tied up to floating docks.

A salt pond on Cape Cod is connected to the ocean by means of a narrow channel. This problem will explore how the water level in the pond varies.

In roughest terms, the water level in the bay increases, over a small time interval, by an amount which is proportional to (1) the difference between the ocean level and the bay level and (2) the length of the small time interval.

Write $y(t)$ for the height of the ocean, measured against some zero mark, and $x(t)$ for the height of the bay, measured against the same mark.

Set up the first order linear equation that describes this model. What is the "system" here? What part of the ODE represents it? What function is the "input signal"? What is the "output signal"?

(c) Suppose now that the ocean height is given by $y(t) = \cos(\omega t)$ (in meters and hours). What value does ω take? (To answer this, assume that the tide is high exactly every 4π hours—not a bad approximation.) Reconcile your equation with the equation that headlines the Mathlet `Amplitude and Phase: First Order`.

(d) It is observed that at its highest, the water level in the bay is $1/\sqrt{2}$ meters above the zero mark. You can model this on the Mathlet! What is the constant called k in the Mathlet? (Solve for it analytically.) What is the time lag? Does your computation match what the Mathlet shows? Write down the steady state solution in the form $A\cos(\omega t - \phi)$.

Problem 2

What is the response of $x' + kx = kq_e(t)$ to the oscillating physical input $q_e = cos\omega t$? (For the temperature model, think of a summer cabin- the evternal temperature rises and falls with a 24-hour period; how does the terperature in the cabin vary (no fireplace and no air-conditioning)?)

The steady-state response looks like $Acos(\omega t - \phi)$. Do the amplitude A and the phase lag ϕ increase or decrease when

(i) you increase k (the conductivity of the walls)?

(ii) the Deity increase ω (the frequency of the earth's rotation)?

(a) First, answer based on your intuition.

(b) Then look at the figlet, move the slidersm and answer based on what you observe.

(c) Without actually solving the equation explicitly, explain why the input curve and the response curve intersect at maximum and minimum points of the response curve.