

# Amplitude and Phase: Second Order

## Problem 1

This problem will use the Mathlet **Amplitude and Phase: Second Order**. Open the applet and play around with it. Animate the spring system by pressing the [ $\gg$ ] key. Chose various values of  $b$ ,  $k$ , and  $\omega$ , and watch the results.

### Justification of the equation written at the top of the screen.

The driving force,  $F_{\text{ext}}$ , is provided by motion of the plunger at the top. Suppose that the length of the spring when it is relaxed is  $d$ . Measure the position of the plunger with reference to a chosen zero point, and measure the position of the mass with reference to a zero point which is  $d$  units lower. This means that  $x = y$  when the spring is relaxed, exerting no force. Now the force on the mass is given by  $-b\dot{x} - k(y - x)$ , so, if we assume that  $m = 1$ , we have the equation  $\ddot{x} + b\dot{x} + kx = ky$ . This Mathlet pictures the situation in which the plunger moves sinusoidally with amplitude 1, centered around  $y = 0$ , and it sets the clock so that  $y = \cos(\omega t)$ .

(a) There is a white line segment on the main window, joining a blue diamond to a yellow diamond. What does its length signify? What does it mean when the blue dot is above the yellow one? Below? when they coincide?

Set  $k = 4.00$  and  $b = 0.50$ . Slide  $\omega$  through its range of values and watch how the system response varies.

(b) For what value of  $\omega$  is the gain, or the amplitude of the system response, the greatest? You can discriminate better among the top values by clicking the [Bode and Nyquist Plots] button and then rolling the cursor over the top right window. This causes a yellow readout giving the amplitude.

The three new windows plot the gain and the phase lag of the system response as functions of the circular frequency of the input signal  $\omega$ . (Actually  $-\phi$  is graphed rather than  $\phi$ .) Verify for yourself that for various values of  $\omega$  the values of  $A$  and  $\phi$  look right for the displayed graph of  $x$ . You can get numerical readouts by positioning the cursor over either of these windows.

The lower right window displays the complex number  $k/p(i\omega)$ . The Exponential Response Formula shows that  $\ddot{z} + b\dot{z} + kz = ke^{i\omega t}$  has exponential solution  $z_p = (k/p(i\omega))e^{i\omega t}$  where  $p(s) = s^2 + bs + k$  is the characteristic polynomial. The the modulus of  $k/p(i\omega)$  is the amplitude of the sinusoidal system response (the “gain”) and the argument of  $k/p(i\omega)$  is the negative of the phase lag.

(c) When  $\omega$  is set at the resonant peak which you found in (b), the phase lag seems to be very close to  $\pi/2 = 90^\circ$ . What are the actual values of  $\omega$  and  $A$  when the gain is maximal? Since the amplitude peak represents “resonance,” let’s write  $\omega_r$  and  $A(\omega_r)$  for these values. The first step will be to differentiate  $f(\omega) = |p(i\omega)|^2$  with respect to  $\omega$ . You’ll need a calculator to get an approximate value (to 3 decimal places) of  $\omega_r$ .

(d) Explain why the phase lag is exactly  $\pi/2$  just when  $p(i\omega) = 0$ , and calculate what  $\omega$  must be for this to occur in the general case  $\ddot{x} + b\dot{x} + kx = k \cos(\omega t)$ . Does

this value of  $\omega$  coincide with the value giving the largest gain in the example we have been studying?

### Problem 2 [Pole diagram]

We consider the spring/mass/dashpot system modeled by the operator  $p(D) = 2D^2 + D + (25/4)I$ : so the mass is 2, the damping constant is 1, and the spring constant is  $25/4$ .

(a) What is the natural circular frequency  $\omega_n$  of this system? Is the system overdamped, underdamped, or critically damped? If it is underdamped, what is the damped circular frequency  $\omega_d$ ? Write down two independent real system responses to the null signal.

(b) We will drive the system through the spring, as in **Amplitude and Phase: Second Order**, so if the position of the top of the spring is given by  $f(t)$ , the relevant equation is  $p(D)x = (25/4)f(t)$ . The transfer function  $W(s)$  is the function of the complex number  $s$  such that for any fixed complex number  $r$ ,  $x = W(r)e^{rt}$  is the exponential system response to the physical input signal  $e^{rt}$ , that is, it is a solution to  $p(D)x = (25/4)e^{rt}$ . (i) Find  $W(s)$ . (ii) Sketch the pole diagram of  $W(s)$ : that is, mark the positions of the poles of  $W(s)$  on the complex plane. (iii) Sketch the graph of  $|W(s)|$ ; this is a surface lying over the complex plane (which we think of as the floor).

(c) Now we consider the frequency response of this system. (i) What is the complex gain  $W(i\omega)$ ? (ii) What is the amplitude of the periodic solution to  $p(D)x = (25/4)\cos(\omega t)$ , as a function of the input circular frequency  $\omega$ ? (iii) Open the **Mathlet Amplitude and Phase: Second Order** and set the sliders appropriately to see this system. (You have to divide through by the mass.) Invoke the “Bode plots” and make sketches of them, graphing the amplitude and the negative of the phase lag of the sinusoidal system response as a function of  $\omega$ . We may allow negative values of  $\omega$  if we wish; then the amplitude is an even function of  $\omega$ . Extend your sketch of the amplitude as a function of  $\omega$  to negative values of  $\omega$ .

(d) Make a new sketch of the graph of  $|W(s)|$ , set up so you can draw the vertical plane meeting the imaginary axis of the complex plane. Draw in the curve where this plane meets the graph of  $|W(s)|$ . Compare this sketch with the sketch you made of amplitude as a function of  $\omega$ , and make a statement about this comparison. Comment on the relationship between the location of the poles of  $W(s)$  and the near resonant peaks in the frequency response.

### Problem 3

In a spring-mass-dashpot system with damping  $b \neq 0$ , suppose in some system of units that the mass and spring constants are  $m = 1$  and  $k = 4$ , and there is a driving force with frequency  $\omega$ , so the differential equation is

$$x'' + bx' + 4x = 4\cos\omega t.$$

Which driving frequency will produce a periodic response with the biggest amplitude?

(a) Answer experimentally with the value  $b = .6 + 0.01N$ , where  $N$  is the recitation

number. (Blue graph is input, yellow is response; note the helpful thin vertical line at the first maximum point.)

**(b)** By calculation, use complex exponentials to find a real particular solution  $x_p$  which is a pure oscillation, and determine its amplitude (use a general  $b$  throughout, not the particular value in part **(a)**).

**(c)** For a given value of  $b$ , for what driving frequency  $\omega$  will the amplitude of  $x_p$  be greatest (this is the “resonant ” frequency)? (Use a blend of calculus and common sense: set  $u = \omega^2$ ; what  $u > 0$  will make the denominator smallest?)

**(d)** Apply your answer in **(c)** to the value of  $b$  you used in part **(a)**, to get the resonant frequency for the ODE of part **(a)**, and compare with the value you found using the figlet.

#### **Problem 4**

Open the applet “Amplitude and Phase: Second Order”. Turn on the Bode and Nyquist plots. Set  $k = 1$ .

**(a)** For each of the following say whether or not there is practical resonance. If there is, give the resonant frequency.

(i)  $b = .5$ . For this one also include a hand copy of the amplitude response.

(ii)  $b = 1.5$ .

**(b)** Compare your results in part **(a)** to the formula.

**(c)** Still with  $k = 1$ , find the value of  $b$  where practical resonance disappears.