Complex Exponential

Problem

(a) Find an expression for \(\sin(4t)\) in terms of sums of powers of \(\sin t\) and \(\cos t\) by using \((e^{it})^4 = e^{4it}\) and Euler’s formula.

The Mathlet Complex Exponential will probably be useful in understanding the rest of this problem. Open it and explore its functionalities. The Help button lists most of them. Notice that in the left window, the real part \(a\) ranges between \(-1\) and \(1\), while the imaginary part \(b\) ranges from \(-8\) to \(8\). You use the left-hand window to pick out a complex number \(a + bi\). When you do, a portion of the line through it and zero is drawn. This line is parametrized by \((a + bi)t\). At the same time, the curve parametrized by the complex-valued function \(e^{(a+bi)t}\) is drawn on the right window.

(b) Sketch the function \(f(t) = e^{-t}\cos(2\pi t)\) for \(t\) between \(-1\) and \(1\). Write down a value of \(a + bi\) such that \(f(t)\) is the real part of \(e^{(a+bi)t}\); sketch a graph of \(e^{(a+bi)t}\) for this value of \(a + bi\); and sketch the curve in the complex plane parametrized by this complex-valued function.

For each of the following questions, explain your answer using Euler’s formula \(e^{(a+bi)t} = e^{at}(\cos(bt) + i \sin(bt))\).

(c) For what values of \(a + bi\) is the curve \(e^{(a+bi)t}\) a circle?

(d) For what values of \(a + bi\) is the curve \(e^{(a+bi)t}\) a ray? What rays are possible?

(e) For what values of \(a + bi\) does the curve \(e^{(a+bi)t}\) converge towards zero as \(t\) grows?

(f) For what values of \(a + bi\) is the curve \(e^{(a+bi)t}\) a spiral which moves away from the origin and curls counterclockwise as \(t\) increases?

(g) Drop the cursor at a nonzero complex number \(a + bi\) and look at the trajectory of \(e^{(a+bi)t}\). (The right plane shows only the part of this curve which corresponds to values of \((a + bi)t\) which fit on the left plane.) For what other values of \(z\) is the trajectory of \(e^{zt}\) the same as the trajectory of \(e^{(a+bi)t}\)?