Convolution: Flip and Drag

Problem 1

(a) What is the LTI operator $p(D)$ with weight function $\sin(t)$ (for $t > 0$)? For this operator, solve the ODE $p(D)x = \sin(t)$ with rest initial conditions by using the Exponential Response Formula (or the Resonant Response Formula if necesssary).

(b) Now solve $p(D)x = \sin(t)$ with rest initial conditions by evaluating the convolution integral $\sin(t) * \sin(t)$.

Open the Mathlet `Convolution: Flip and Drag`. This is a popular and useful way of thinking of the convolution integral. The input signal is called $f(t)$ (and it's red). The intermediate time variable is called u (rather than τ). The weight function is called $g(t)$ (and it's green). Accept the default choices $f(t) = \sin(t)$, $g(t) = e^{-t}$. Adjust the time slider so $t = 8.00$.

The perspective here is that the value of the convolution at $t = 8.00$ is obtained by integrating $f(u)$ as u ranges from $u = 0$ to $u = t$; but the values have to be weighted appropriately. The weight function here is e^{-t}, so the contribution of $f(u)$ to the value of the integral isn't $f(u)$, but rather $f(u)e^{t-u}$. In general it's $f(u)g(t-u)$.

The graph of $g(t-u)$ (for t fixed and u varying) is the graph of $g(u)$ "flipped" (across the vertical axis) and "dragged" to the right by t units. This is drawn in green on the bottom left window. The window at middle left graphs the product of $f(u)$ with $g(t-u)$ (for fixed t). The convolution integral is the integral of that product, i.e. the signed area under the curve. That area is shaded in cyan, and graphed in the top window.

To get a feel for how this works, position t back at -1 and click the `[>>]` button. Notice how the influence of the signal at a given time decreases as time goes on.

Now select $g(t) = \sin(t)$.

(c) Explain as well as you can, in words, how the Mathlet illustrates the phenomenon of resonance.

(d) At what values of t do you expect the maxima of $\sin(t) * \sin(t)$ to occur, on the basis of this simulation? Verify that this is correct, from your work in (i)– (ii).

Problem 2

This figlet gives a helpful geometric picture of the convolution $f(t) * g(t)$. Each of the three areas graphs the functions in its header. A slider at the bottom sets the t-value.

Try playing around with different $f(u)$ and $g(u)$. Recommended choices are $f(t)$ the unit box function on [0,3] and $g(u) = e^{-u}$; $f(u) = sin2u$ and $g(u) = sinu$.

If you prefer, try simpler functions first. Study the effect on the three graphs as you move the t-slider.

(a) (i) In the middle graph, what is the significance of the shading?

(ii) In the bottom graph, fix a value of t: how is the resulting green graph related geometrically to the graph of $g(u)$? From this, derive the formula $g(t-u)$ for its function.

(b) Take both f, g to be $sinu$.

(i) How do the graphs of $f(u)$ and $g(t-u)$ look when t is at a maximum point of $(f * g)(t)$? From this, determine without calculation all the maximum points $t > 0$ of $(f * g)(t)$;

(ii) Without calculating $f * g$ explicitly in terms of t, use your answer to (i) to give the value of $f * g$ at its nth maximum point.

(c) Calculate $f * g$ explicitly using the Laplace transform and tables, and from this find its maximum points and verify your answer to (b, (ii)).

(d) Show that $sint * cost$ is the solution to $y'' + y = sint, y(0) = y'(0) = 0$, and explain briefly the connection between resonance and your answer to (b, (ii)).