

# Convolution: Flip and Drag

## Problem 1

(a) What is the LTI operator  $p(D)$  with weight function  $\sin(t)$  (for  $t > 0$ )? For this operator, solve the ODE  $p(D)x = \sin(t)$  with rest initial conditions by using the Exponential Response Formula (or the Resonant Response Formula if necessary).

(b) Now solve  $p(D)x = \sin(t)$  with rest initial conditions by evaluating the convolution integral  $\sin(t) * \sin(t)$ .

Open the Mathlet **Convolution: Flip and Drag**. This is a popular and useful way of thinking of the convolution integral. The input signal is called  $f(t)$  (and it's red). The intermediate time variable is called  $u$  (rather than  $\tau$ ). The weight function is called  $g(t)$  (and it's green). Accept the default choices  $f(t) = \sin(t)$ ,  $g(t) = e^{-t}$ . Adjust the time slider so  $t = 8.00$ .

The perspective here is that the value of the convolution at  $t = 8.00$  is obtained by integrating  $f(u)$  as  $u$  ranges from  $u = 0$  to  $u = t$ ; but the values have to be weighted appropriately. The weight function here is  $e^{-t}$ , so the contribution of  $f(u)$  to the value of the integral isn't  $f(u)$ , but rather  $f(u)e^{t-u}$ . In general it's  $f(u)g(t-u)$ .

The graph of  $g(t-u)$  (for  $t$  fixed and  $u$  varying) is the graph of  $g(u)$  “flipped” (across the vertical axis) and “dragged” to the right by  $t$  units. This is drawn in green on the bottom left window. The window at middle left graphs the product of  $f(u)$  with  $g(t-u)$  (for fixed  $t$ ). The convolution integral is the integral of that product, i.e. the signed area under the curve. That area is shaded in cyan, and graphed in the top window.

To get a feel for how this works, position  $t$  back at  $-1$  and click the [>>] button. Notice how the influence of the signal at a given time decreases as time goes on.

Now select  $g(t) = \sin(t)$ .

(c) Explain as well as you can, in words, how the Mathlet illustrates the phenomenon of resonance.

(d) At what values of  $t$  do you expect the maxima of  $\sin(t) * \sin(t)$  to occur, on the basis of this simulation? Verify that this is correct, from your work in (i)– (ii).

## Problem 2

This figlet gives a helpful geometric picture of the convolution  $f(t) * g(t)$ . Each of the three areas graphs the functions in its header. A slider at the bottom sets the  $t$ -value.

Try playing around with different  $f(u)$  and  $g(u)$ . Recommended choices are  $f(t)$  the unit box function on  $[0,3]$  and  $g(u) = e^{-u}$ ;  $f(u) = \sin 2u$  and  $g(u) = \sin u$ .

If you prefer, try simpler functions first. Study the effect on the three graphs as you move the  $t$ -slider.

(a) (i) In the middle graph, what is the significance of the shading?

(ii) In the bottom graph, fix a value of  $t$ : how is the resulting green graph related geometrically to the graph of  $g(u)$ ? From this, derive the formula  $g(t-u)$  for its function.

**(b)** Take both  $f, g$  to be  $\sin u$ .

(i) How do the graphs of  $f(u)$  and  $g(t - u)$  look when  $t$  is at a maximum point of  $(f * g)(t)$ ? From this, determine without calculation all the maximum points  $t > 0$  of  $(f * g)(t)$ ;

(ii) Without calculating  $f * g$  explicitly in terms of  $t$ , use your answer to (i) to give the value of  $f * g$  at its  $n$ th maximum point.

**(c)** Calculate  $f * g$  explicitly using the Laplace transform and tables, and from this find its maximum points and verify your answer to (b, (ii)).

**(d)** Show that  $\sin t * \cos t$  is the solution to  $y'' + y = \sin t, y(0) = y'(0) = 0$ , and explain briefly the connection between resonance and your answer to (b, (ii)).