Euler 's Methods

Problem 1

(a) The solution to $y' = y$ with $y(0) = e^x$. Compute y_k for $k = 0, \cdots, 3$ for n fixed but arbitrary. What is the Euler approximation for $e = y(1)$, using n equal steps?

The rest of this problem will use the Euler's Method applet to investigate the dependence of the error of Euler's method on the step size. Invoke Euler's Method and familiarize yourself with it for a few minutes. It illustrates Euler's method applied to the differential equation $y' = f(x, y)$, where $y' = dy/dx$. Select $f(x, y) = ysinx$, and carefully select the initial condition $(x, y) = (0.00, 1.00)$. Use the tool to obtain the Euler's method estimates of $y(1)$, for the various step sizes available, and also the "actual" value.

(b) Make a table with the following columns: the step size h, the Euler estimate using this step size, the "error" corresponding to h (that is, the actual value minus the estimated value), and the error divided by h. Does your table support the claim that the error is approximately proportional to h? For comparision, make a table of the error divided by h^2. Which is closer to constant? [The "improved Euler method" or RK2, would lead to errors roughly proportional to h^2, RK4 would give errors roughly proportional to h^4.]

(c) Is the estimated value larger or smaller than the actual value? Please explain why this is, in terms of the direction field.

(d) Find the solution to $y' = ysinx$ with $y(0) = 1$ analytically, and use a calculator to find $y(1)$. Does it coincide with the "Actual" value given by the Mathlet?

Problem 2

(a) To get familiar with the figlet, call up $dy/dx = y$, take $y(0) = .2$ as the initial condition, and use step- size $h = .5$. Using "next step" repeatedly, construct step-by-step the Euler method solution $y(t)$, (using $h = .5$) and report $y(2.5)$. Repeat, but this time using $h = .25$. Report these two values, along with the correct value, obtained by solving the equation.

(b) This studies how the error in Euler's method varies with the step-size. Call up $dy/dx = ysinx$, select All Euler, use it to draw in all the Euler solutions satisfying

$$y(-3.90) = 1.00 + (.03) \times (\text{your recitation number}).$$

Using the mouse, obtain the value of $y(3.90)$ for each step-size. Comparing with the "actual" solution, calculate for each Euler- step-size h the corresponding error e in $y(3.90)$, and plot e against h; draw by eye the best line through these data points, and write an approximate equation $e = ch$, using it, tell what Euler step-size should give the correct value of $y(3.90)$ to two decimal places.

(c) As a final cautionary tale, call up $dy/dx = y^2 - x$, use as the initial point (-0.98,0), and construct the Euler solution using $h = .5$, for increasing x. Then use the same initial condition with All Euler. Report what happend, and what caused the difficulty. (Numerical work in Ordinary Differential Equation has many such pitfalls.)

Problem 3

(a) Call the equation $y' = .5y + 1$. Use the cursor to select an initial value and a step-size, and use the "start/next step" key to click out the corresponding Euler polygon. Do this for several choices of step-size. The polygons will be neatly stacked up. Can you explain why this is? In other words, Euler's method gives an approximate solution which is (as time increases) too small. Why?

Now, invoke the equation $y' = ysinx$ and set the initial condition to $(x_0, y_0) = (-4, 1)$. Select step-size $h = 1$ and click out a solution to the right edge of the screen. You can read off the value of the approximation at $x = 4$ from the table at left. Make a note of it. Then do the same for $h = 1/2$, $h = 1/4$, $h = 1/8$, and finally "actual." When the step-size gets small, there are too many steps for the table to record, and you will have to measure the value of the approximation at $x = 4$ using the cursor. Make the graph plotting the step-size horizontally and the error at $t = 4$ vertically. Fit this by eye with a straight line. What is its slope? This is the number m for which the erro E at $x = 4$ for step-size h is approximately mh. How many steps would you need to approximate $y(4)$ with an error of at most 0.1 using Euler's method?

[If you were to make the same graph of error vs step-size for "Heun 's method", also known as the improved Euler method, you would get a quadratic relationship: $E \simeq kh^2$. Using RK4 you'd get $E \simeq kh^4$. This is why these three methods are called, respectively, first, second, and fourth order. In fact, there are Runge-Kutta methods of all orders, and Euler's method is RK1 and Heun's is in RK2.]

(b) If you invoke Euler's Method with $f(x) = -xy$, and poke around, you will see that for many values of the step-size $\triangle x$ and many choices of initial condition (x_0, y_0). Euler's method goes bananas; it oscillates more and more widely around $y = 0$, which, pretty clearly, is a solution to which all other solutions converge. Verify this observation: Identify the conditions on $\triangle x, x_0, y_0$, which are required for Euler's method to successfully approximate a solution for large time. Hint: notice that $y_{n+1} = y_n(1 - x_n \triangle x)$.

Problem 3

Consider the equation $y' = ysinx$ with initial condition $y(-3.8) = -1.1$.

(a) Approximate $y(-3.8 + \frac{k}{8})$ for $k = 1, 2, 3, 4$ using the Euler method with step size $h = 0.125$. Gove results to 4 decimal places.

(b) Solve the given equation to get a formula for $y(x)$, and then use that formula and a computer or calculator to get $y(-3.8 + \frac{k}{8})$ 'exact' to 6 decimal places for $k = 1, 2, 3, 4, 8, 10, 20$.

(c) We will use the computer visual applet called "Euler 's Method". Carefully position the pointer with the mouse to get the initial condition $y(-3.8) = -1.1$. Then use the "Start/Next Step" button to run Euler's method with step size $h = 0.125$ as far as $t = -1.30$, and record the results at $t = -3.8 + \frac{k}{8}$ for $k = 1, 2, 3, 4, 8, 10, 20$.