Fourier Coefficient

Problem 1

This problem will use the Mathlet `Fourier Coefficients`

When the applet opens you are presented with a series of sliders labeled b_n. By pressing the [`Formula`] radio button you can see that they are coefficients of sines in a Fourier series made up entirely of sine functions. If you press the [`Cosine`] radio button you'll see a_n's, coefficients of cosines.

Move one of the slider handles: a cosine or sine curve appears and changes amplitude. Release it at some value and move another one. The white curves shows the new sinusoid, and the yellow curve shows the sum of the two. By moving more sliders you can build up more complicated sums and more complicated functions.

Now select the `Target` [`D`]. Is it an even function or an odd function? Based on this, decide whether to appoximate it using sines or cosines. Select one or the other appropriately (using [`All terms`]) and do the best you can by eyeballing the result to get the best approximation you can to the green target curve.

(a) Write down these values of the coefficients.

(b) The target function D is the periodic function with period 2π which is given by $f(t) = t$ for $-\pi < t < \pi$. Compute the Fourier coefficients for this function, using the integral formulas for them, and compare with your answers from **(a)**. Reset the sliders to the computed values and see if it looks like a better fit.

(c) Now select target function A. It is an odd function so its Fourier series will only involve sines. Select [`Sine`] and [`All terms`]. Move one of the even sliders. Explain in words why Fourier terms of the form $b_{2n} \sin(2nt)$ don't look like they'll be useful. Then select [`Odd terms`]. Instead of eyeballing the fit as before, select the [`Distance`] button. The number that appears at the upper right hand corner is the "root mean square" distance between the green target function and the yellow Fourier approximation. It's a measure of goodness of fit. Pick values for $b_1 \ldots b_6$ at random. Then start from the bottom and successively adjust the sliders to minimize the distance. Write down the optimal values of the b_n's. Now hit [`Reset`] and do the same thing starting from the top. Again write down the optimal values. Finally, using the fact that this function is $(\pi/4)(t)$, write down what the Fourier coeffients actually are and compare with your two answers.

Lessons: (1) The Fourier coefficients are the coefficients resulting in the best possible fit, and **(2)** the process of optimizing one coefficient is independent of the process of optimizing any of the others.

(d) Using Lesson **(1)**, what is the Fourier series for $\sin(t - \pi/4)$?

(e) Finally, for each of the remaining target functions B, C, E, and F, decide whether you need to use sines and cosines, and given that choice whether you have to use all terms, odd terms, or even terms. Make a table with columns: Letter [A-F, including A and D]; Even/Odd; Right half even/odd; Even/Odd; Sine/Cosine. By "Right half even/odd" I mean: is the right half of the graph even or odd about its midpoint; i.e. is $f(t - \pi/2) = \pm f(t + \pi/2)$? In each case the answer might be "neither."

Problem 2

(a) Using the function E, find the best values you can for the coefficients of an approximating F. sine series; record the value and the distance in a table.

Compare with the results of the formula; the scale on the axes in the figlet will tell you what L should be.

(b) To get further experience with F. approximations, seeing how each successive term improves the approximation, find the best approximations for the functions C and F. (Use a Fourier sine series for an odd function, a Fourier cosine series for an even function. By using the distance, find the best value for each successive coefficient, and watch the yellow approximating graph. At the end, give the F. approximation, with your numerical values for the coefficients, and record also the distance from $f(t)$.)

(c) For each of the functions E and F give

(i) The rate of decay of the Fourier coefficients (i.e. what power of n is in the denominator).

(ii) Give the smoothness of each of the functions-i.e. do they have discontinuity, a corner or are they smooth?

(iii) Visually, which Fourier series converges faster according to the applet- that is, as you optimize each coefficient in increasing order, which series is closer to the function it's approximating?

(iv) Suppose we looked at a smooth periodic function (i.e. with continuous first derivative). Make a conjecture about hte rate of decay of its Fourier coefficients.

Problem 3

(a) Compute the Fourier series of the function which is periodic of period 2π and is given by $f(t) = t/2$ for $-\pi < t < \pi$, by evaluating the integral expressions for a_n, b_n (or using even/odd to decide that one or another of these sets of numbers are all zero).

(b) Select target A. Move the sliders around till you get the best fit you can eyeball. Record your results.

Now, select target D and do the same. But then, before you record your rresults, select "Distance." This makes a number appear above the graph, which gives a measure of the goodness of fit of the partial Fourier series you have built. Move the sliders from the top one to the bottom one to get the best fit you can. Record the results. Notice that you began with large period and then worked your way down to small period.

Now press "Reset", and do the same thing from the bottom up: you are putting in the best possible multiples of $sin6t$, then $sin5t$, and so on, in that order. Are the numbers you obtain the same as the ones you got going in the other direction? How do these values match up with what you computed in Part (**a**)? Do you suppose you would get different answers if you put interms in some other more random order?

Finallym for each of the other targets declare which option among "Sine Series" and "Cosine Series," and among "All terms," "Odd terms," and "Even terms," gives the best fit.