Linear Phase Portraits: Matrix Entry

Problem 1 [Linear system; the companion matrix]

(a) We'll work with the two homogeneous constant coefficient linear equations $\ddot{x} + 3\dot{x} + 2x = 0$ and $\ddot{x} + 2\dot{x} + 4x = 0$. For each, find two independent real solutions, (please use either exponentials or functions of the form $e^{rt}\cos(\omega t)$ or $e^{rt}\sin(\omega t)$, and denote them by $x_1(t)$ and $x_2(t)$), write down the general real solution, and determine the damping characteristic. Also compute $\dot{x}_1$ and $\dot{x}_2$.

(b) Now write down the companion matrix for each of these two equations. This means: set $y = \dot{x}$ and then solve for $\dot{y}$ in terms of x and y, to get a system of two linear equations, of the form $= A$, where $= xy$ and $A = abcd$. In a companion matrix $a = 0$ and $b = 1$.

(c) Open the Mathlet `Linear Phase Portraits: Matrix Entry`. Select "Companion Matrix," and set the c and d values to the entries of the companion matrix for the first equation. (Note that clicking on a hashmark on a slider sets the value.)

For a companion matrix $A = 01cd$, the colorful window at the upper left shows $(d, -c)$. For the meaning of this in terms of the damping conditions of the second order equation, see the notes to Lecture 13.

The big window shows the "phase plane" of the system. It displays the trajectories of a few solutions. Click on the window to produce more. You can clear them all using `[Clear]`, and return to the original set of trajectories by re-setting one of the c or d sliders. Do this; return to the originally displayed selection of trajectories.

Since $y = \dot{x}$, a solution to $= A$ is given by $x(t)\dot{x}(t)$ where $x(t)$ is a solution of $\ddot{x} + 3\dot{x} + 2x = 0$ (in this first case). Draw a picture of the phase plane. Each of these trajectory curves should have an arrow on it indicating the direction of time: please indicate this on your picture. Identify which of the trajectories correspond to each of the basic solutions you found in **(a)**. (These will be among the originally chosen trajectories.)

(d) There is a hook-shaped trajectory in the upper half plane. The picture doesn't show a scale; but suppose that it crosses the y axis at $(0, 1)$. What is the solution having this as its trajectory assuming that this crossing occurs at $t = 0$?

(e) Write down another solution having the same trajectory. (There are infinitely many!)

(f) Now set the c and d sliders to the values relevant to the second equation you solved in **(a)**. Sketch the phase portrait (and include the arrows indicating the direction of time). The picture doesn't show a scale; but suppose that one of the shown trajectories crosses the y axis at $(0, 1)$. What is the solution having this as its trajectory assuming that this crossing occurs at $t = 0$. At what times does this solution cross the y axis in the future? Sketch, roughly, the graphs of $x(t)$ and of $y(t)$.

Problem 2 [Eigenvalues, eigenvectors]

(a) Find the eigenvalues and eigenvectors of the companion matrix for $\ddot{x}+3\dot{x}+2x = 0$. On the x, y plane draw the eigenlines. For each of the two eigenlines, write down a solution which moves along it. Compare this with the work you did in **31.**, especially in part **(c)**.

(b) Write down the companion matrix for the equation $\ddot{x} + 2\dot{x} - 2x = 0$. Find the eigenvalues and eigenvectors for this matrix, and sketch the eigenlines.

Now, invoke `Linear Phase Portraits: Matrix Entry`, set c and d to display the phase plane for this companion matrix, and sketch the phase plane that it displays. Include arrows indicating the direction of time.

For each of the eigenlines, write down a solution that moves along it.

Problem 3 [Qualitative behavior of linear system]

Invoke the Mathlet `Linear Phase Portraits: Matrix Entry`. Play with the tool for a while to get a feel of it. Notice that the eigenvalues can be displayed on the complex plane. Deselect the `[Companion Matrix` option, so you can set all four entries in the matrix. Select the `[eigenvalues]` option, so the eigenvalues become visible by means of a plot of their location in the complex plane and also a read-out of their values.

We will use this tool to investigate the phase portraits of the homogeneous linear equation $= A$, where $A = 13 - 1d$, as d varies. To start with, set the matrix to $A = 13 - 1 - 4$. Then move the d slider up to $d = 4$, and watch (1) the movement of the mark on the (Tr,Det) plane; (2) the movement of the eigenvalues in the complex plane; and (3) the variation of the vector field.

(a) Compute the trace and determinant of A. (They will depend upon a, of course.) Find an equation for the curve (or line) traced out by the mark on the (Tr,Det) plane.

(b) You notice that the curve in the (Tr,Det) plane enters a number of different regions. When the cursor crosses a red boundary, the trajectories and the eigenvalue indicators turn red. Work out what the values of d are at those crossings. (So this is: where $\det A = 0$, where $\det A = (A/2)^2$ (twice, once not represented on the Mathlet), and where $A = 0$.

(c) There are nine phase portrait types represented as d varies (five regions and four walls). Draw an interval from -4 to $+5$. On it, mark the four values of d at which the matrix crosses one of the walls. Indicate the type of phase portrait you have at each of the marked points and along the intervals between them. That is, classify the phase portrait into one of the following types, as in the Supplementary Notes, §25: spiral (stable/unstable, clockwise/counterclockwise), node (stable/unstable); saddle; center (clockwise/counterclockwise); star (stable/unstable); defective node (stable/unstable; clockwise/counterclockwise); degenerate (comb (stable/unstable), constant, parallel lines).

(d) For each of the four special values, and for your choice of one value in each of the five regons, make a sketch of the phase portrait. Be sure to include and mark as such any eigenlines, and the direction of time.

(e) Letting $T = tr(A)$ and $D = det(A)$,

Give the equation of the parabolic red line in the picture;

For the left-hand yellow and green religions, and the blue region below, show algebraically that the behavior of the eigenvalues of A above implies the location of the corresponding point (T, D).

(f) There are spirals and spirals-some are "loose", going around many times as they approach the origin; others are "tight", with all the spiralling done so close to the origin that one would need a huge zoom-in magnification to see it. By experimenting with different points in the "spiral sink" region of the TD- screen. Determine the connection between the complex eigenvalue $\lambda = r + si$ and the tightness of the spiral and explain it in a few words mathematically.

(g) In a node or a saddle picture, the eigenvectors will lie along the two straight line trajectories going through the origin. By experimenting with the figlet in the node and saddle pictures, guess what the form of the matrix A should be if the perpendicular vectors $\vec{i} \pm \vec{j}$ are both eigenvectors. Prove using algebra that your form for the matrix A is correct.

Problem 4

Let c be a real constant. This problem will analyze the system

$$x' = y$$

$$y' = cx - 2y$$

(a) What is the characteristic polynomial of A the coefficient matrix for the system?

(b) Compute its eigenvalues and eigenvectors. The answer depends on c so you will need to break your answer into cases.

(c) Write down the general solution to the equation. Again you will need to break into cases depending on c.

(d) Open the visual Linear Phase Portraits: Matrix Entry and click on the eigenvalues button. Using representative values of c give sketches of all the different types of phase portraits possible as c varies. Using your answer in part **(c)** explain the portrait when $c = -3$.