Phase Lines

Problem 1

This problem will use the Mathlet `Phase Lines`. Open the applet and understand its use and conventions. Click on [Phase Line] to see a representation of the phase line. Note the color coding: a green dot represents a stable or attracting equilibrium; red represents an unstable or repelling equilibrium; and blue represents a "semi-stable" equilibrium.

The autonomous ODE $\dot{y} = .25 - ay + y^2$ models a population of Australian lovebugs, which infest pomagranites in Farmer Jones's orchard. y is measured in megabugs, or millions of bugs. The term .25 reflects a constant immigration into Mr Jones's orchard from the neighboring orchards (where the pomagranites are inferior). These pests can be kept in check using an expensive bioengineered spray. Application at a rate a moves the natural rate of growth of the lovebug population from y (corresponding to $\dot{y} = -y^2$) down to $y - a$ (corresponding to $\dot{y} = (y - a)y$).

(a) Use the Mathlet to determine (approximately) the smallest rate a which will contain the lovebug population at a finite level. What is that level? Now check this answer analytically. Why is this level of application a dangerous strategy for Farmer Jones?

(b) Better will be a choice of a which brings the lovebug population down to $y = 0.25$. What rate a will lead to that result, according to the Mathlet? How large an initial population will this rate of application control? Now check these answers analytically.

(c) For this value of a, there are five different behaviors possible for the lovebug population. Two solutions exhibit the "same behavior" if one is a time-translate of the other. Sketch one solution of each of the five types. Your sketch should make it clear what the behavior of the solution is as $t \to -\infty$ and as $t \to \infty$.

(d) Invoke the Bifurcation Diagram for this autonomous equation. Move a along its slider to see the variety of behaviors the phase line of $\dot{y} = .25 - ay + y^2$ as a varies. The green and red curve in the newly displayed bifurcation plane represents the equilibrium points for those equations, for various values of a. Give an equation for that curve.

Problem 2

(a) Which of the six ODEs on the menu are linear?

(b) Select the equation $y' = ay + y^3$ and click on the Bifurcation Diagram box. Move the a slider to see how the phase lines for various values of a fit into the bifurcation diagram. THe bifurcation diagram is a curve in the (a, y) plane. What is an equation for this curve?

(c) Now select $y' = y^2 + a$ and set $a = -1$. The solutions of $y' = y^2 - 1$ fall into five types, depending upon what the initial condition is. Two solutions are of the same "type" if they are horizontal translates of each other. Sketch a representative of each of the five types.

(d) Representatives of the five types occur with $y(0) = -2, -1, 0, 1, 2$. Solve the ODE and write down explicit formulas for each one of those five solutions. For each one, say what its maximal domain of definition is: two of them do not extend to functions defined for all time. As a preliminary step, solve for y in terms of z if

Problem 3

Let $y = y(t)$ represent a time-varying population size. In this problem we'll examine the behavior over time - and in particular in the long run - of $y(t)$ for two different types of birth-and-death rate situations. One is modeled by the simplest (i.e. linear) DE and the other by a non-linear (but still autonomous) DE.

(a) Suppose that the model being used gives the DE $y' = -ay + 1$. Find the intervals for the parameter a which determine the long-range stability behavior of the population, and illustrate one typical case for each type (i.e. with a representative value of a in each range). Use both (i) the phase line diagram with critical points and (ii) (rough sketches of) the y vs. t graphs.

(b) Repeat part **(a)** for $y' = y(1 - y) - a$.

(c) Describe in words the population models in parts **(a)** and **(b)**.

(d) Open the "Phase Lines" applet, and click the 'Phase Line' button to show the phase line. Now use the applet to check your results for parts (a) and (b). (Note: the color-coding uses green for stable equilibria and red for unstable –look at the yellow arrows.)

Finally, click the 'Bifurcation Diagram' button and experiment until you understand what it is telling you. For each of the cases **(a)** and **(b)** is there a bifurcation point, and if so, what is its significance in terms of the behavior of the population being described in each case?