Amplitude and Phase: Second Order, II

This Mathlet explores a spring/mass/dashpot system driven through the dashpot. It also models a series RLC circuit in which the system response of interest is the voltage drop across the resistor. You can think of it as a primitive AM radio receiver.

Vary the input signal ω, the spring constant k, and the dashpot constant b, and observe the relationship of the system response (the displacement of the mass, represented by the yellow curve) to the input signal (the displacement of the end of the dashpot, represented by the cyan curve).

1. On the basis of this experimentation, answer the following questions.
(a) What is the maximum gain that this system is capable of? (Since the amplitude of the input signal is 1, this is the same as the amplitude of the system response.)
(b) What is the gain at low frequency (say $\omega = 0$)?
(c) When the frequency is such that the gain is maximal – that is, when $\omega = \omega_r$, the resonant frequency – what is the phase lag?
(d) When $\omega < \omega_r$, what can you say about the phase lag?

2. Now invoke the Bode Plots windows. You can verify the observations you just made using it; but you can also start to explore the system response in even more detail, by addressing the following questions.
(a) How does ω_r depend upon the system parameters k and b? Would you like to hazard a guess about an explicit formula?
(b) How does the sharpness of the resonant peak depend upon k and b?

3. Now is the time to verify the observations you made in the first two problems, with pencil and paper. (Question **2 (b)** is deferred to **5.** below.)

4. You observed that there is a strong relationship between the gain and the phase lag; in particular the phase lag is very simple when the gain is maximal. This relationship is vividly portrayed by the "Nyquist plot," which you should now invoke. Move the various sliders and observe what happens. The Nyquist plot is the trajectory of the complex gain, here given by

$$G(\omega) = ib\omega/p(i\omega)\,, \quad p(s) = s^2 + bs + k\,.$$

As usual, the gain is $|G(\omega)|$ and the phase lag is $-\mathrm{Arg}(G(\omega))$. Confirm this relationship using the Mathlet.

(a) Verify algebraically that this trajectory is independent of the system parameters b and k.

(b) Verify algebraically that this curve in the complex plane is in fact what it looks like it is.

5. In terms of the radio receiver, the narrower the resonant peak the better the tuning: frequencies other than those very close to ω_r are suppressed. This can be measured by using the "half power point" $\omega_{1/2}$. This is the frequency at which the gain is about $1/\sqrt{2}$ times the maximal gain. (Power is proportional to the square of the amplitude.) If we write $\omega_{1/2} \simeq \omega_r \pm \epsilon$, estimate ϵ in terms of the system parameters.