Amplitude Response and Pole Diagram

The behavior of an LTI (linear, time invariant) system is determined by its transfer function $W(s)$. In particular, the system response to harmonic excitation—i.e. a sinusoidal input signal, say $B \cos(\omega t)$—is given in terms of the modulus and argument (or amplitude and angle)

$$ g(\omega) = |W(i\omega)| \quad \phi(\omega) = \text{Arg}(W(i\omega)) $$

by

$$ x_p = gB \cos(\omega t - \phi). $$

The Mathlet Amplitude Response and Pole Diagram illustrates this relationship in the case of the system with

$$ W(s) = \frac{k}{ms^2 + bs + k} $$

This models a spring/mass dashpot system driven through the spring, as illustrated in the Mathlet Amplitude and Phase: Second Order I. It is given by a second order linear constant coefficient equation

$$ m\ddot{x} + b\dot{x} + kx = kB \cos(\omega t) $$

In the Mathlet, we set the input signal amplitude $B$ to $B = 1$, and the mass $m$ to $m = 1$.

The modulus of the transfer function, $|W(s)|$, is a real-valued function on the complex plane. As such it can be visualized using its graph, which is a surface lying over the plane, with elevation $|W(s)|$ over the complex number $s$.

The formula $g(\omega) = |W(i\omega)|$ means that the gain (as a function of input circular frequency) is read off by tracing the curve on the graph of $|W(s)|$ lying over the imaginary axis. This curve is the intersection of the graph of $|W(s)|$ with the plane rising vertically from the complex plane and meeting it at the imaginary axis.

The long term behavior of the LTI system with transfer function $W(s)$ can be read off from the divisor or zero/pole locus of $W(s)$. Since sinusoidal solutions are persistent, their general features can be deduced from the divisor of $W(s)$.

This is quite visible from the Mathlet, in this second order example. This transfer function is a rational function with constant numerator, so there are no zeros. The denominator is quadratic (if $m \neq 0$), so there are two poles. (The denominator is the characteristic polynomial of the differential operator, so the poles occur at the characteristic roots.)
When $b$ is not too large relative to $k$, the poles have nonzero imaginary part and negative real part. The graph of $|W(s)|$ sweeps up to $\infty$ as $s$ approaches a pole. The gain curve rises up over the shoulder of the mountain with peak lying over a pole. This accounts for near-resonance. As the damping decreases, the pole real part of the pole decreases; it approaches the imaginary axis, and the gain curve traverses a higher and higher ridge: the resonant peak becomes more pronounced. When the damping becomes zero, the pole comes to rest on the imaginary axis, the gain becomes infinite at that frequency, and we experience resonance.

When the damping term is large relative to the spring constant, the system becomes overdamped. This means that the roots of the characteristic polynomial become real, so the poles of $W(s)$ become real. In this situation, the center of the ridge surmounted by the gain curve is at $\omega = 0$; there is no resonant peak for $\omega > 0$. This collapse of $\omega_r$ to zero as $b$ grows occurs before critical damping, since along the imaginary axis the relative position of two nearby poles is imperceptible.