Eigenvalue Stability Lab

Today we will use an interactive mathlet to understand the concept of eigenvalue stability. Special thanks to Professor Haynes Miller and Jean-Michel Claus for their tireless efforts in putting this mathlet together.

---

NAME_________________________

1. **Open the Eigenvalue Stability Mathlet**

   Open the applet by navigating to
   
   http://math.mit.edu/~jmc/daimp/EigenvalueStability.html
   
   in your favorite web browser. If nothing comes up, you may have to install/enable the Java plugin and relaunch your browser. It should only take a minute.

2. **Stability Boundary**

   (a) Select one of the available numerical integration schemes from the pull-down menu in the lower left, and check the Formula box below. This formula displays the amplification factor $g$ as a function of $z = \lambda \Delta t$.

   (b) Choose two (2) values of $\theta$ ($\theta_1$ and $\theta_2$) in the range $[0, 2\pi)$. These angles define two (2) points on the unit circle in the $g$-plane, $g = e^{i\theta_1}$ and $g = e^{i\theta_2}$. In the space below, calculate the corresponding values of $z$ ($z_1$ and $z_2$).

   $\theta_1 = $ _____________________________ $\theta_2 = $ _____________________________

   $z_1 = $ _____________________________ $z_2 = $ _____________________________
(c) Now grab the \( \theta \) slider in the upper right and, in turn, set it to each value of \( \theta \) you selected above. As you change \( \theta \) you see the yellow diamond in the main figure trace out the corresponding values of \( z \) (i.e., the stability boundary). Check your solution above by rolling over the yellow diamond in the main figure to see its complex coordinates.

(d) Select a different numerical integration scheme. Drag the \( \theta \) slider from \( \theta = 0 \) to \( \theta = 2\pi \) and observe how the stability boundary is traced out in the \( z \)-plane in real time.

3. Eigenvalue Stability

(a) Select either Forward Euler (RK1) or RK2 from the pull-down menu.

(b) Consider the numerical solution of the following parameterized initial value problem:

\[
\frac{d}{dt} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -a & 2a \\ -a & -3a \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad a \neq 0, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \in [0, 10].
\]

Find the eigenvalues as a function of the parameter \( a \).

\[
\lambda_1 = \quad \lambda_2 =
\]

(c) Let \( a = 2\sqrt{5} \). For the method you selected in part (a), use the applet to determine the maximum allowable time step \( \Delta t \) which preserves eigenvalue stability.

\[
\Delta t =
\]

(d) Computational resource restrictions and sufficient accuracy requirements demand that you use the time step \( \Delta t = 0.1 \). Find the range of the parameter \( a \) for which your numerical integration will be stable.

\[
a \in [\quad , \quad ]
\]
4. Pick A Scheme

For each of the following scenarios, suggest a numerical integration scheme from the set

\{Forward Euler (RK1); Midpoint; Backwards Euler; Trapezoidal; RK2\}

and select the time step \(\Delta t\). Provide a brief justification for your selection.

(a) The governing equation is \(u_t = f(u) = -u^3\) with initial condition \(u(0) = 1\). We need second-order accuracy, but evaluations of \(f(u)\) are very expensive.

Method: \(\Delta t = \) ____________

(b) There are competing populations of goldfish \(G(t)\) and sharks \(S(t)\). We have \(dG/dt = G(1 - S)\) and \(dS/dt = (G - 1)S\) with \(G(0) = 1\) and \(S(0) = 1\). We are interested only in the qualitative behavior of the solution (i.e. accuracy is not paramount).

Method: \(\Delta t = \) ____________

(c) The governing equation is for a damped oscillator, \(\theta_{tt} + \theta_t + \theta = 0\) with initial condition \(\theta(0) = \pi/4\). Matrix inverses are disallowed, even in componentwise form. The output of interest is \(\theta(1)\). We can only afford 50 matrix-vector multiplies.

Method: \(\Delta t = \) ____________