Amplitude and Phase: First Order

The model. The Mathlet Amplitude and Phase: First Order illustrates the steady state solution to the first order linear differential equation

$$\dot{x} + kx = k\cos(\omega t)$$

Equations of this type arise in many contexts. Here is one. The level of the ocean, at any particular location, varies in response to the gravity of the moon (and, to a lesser extent, of the sun), producing *tides*. Let's write $y(t)$ for a measure of this water level at the mouth of a channel connecting a lake to the ocean.

Let's write $x(t)$ for the water level in the lake. Arrange these two parameters so that if they are equal then the two are in equilibrium.

Suppose that at $t = a$ the water level in the lake is greater than the water level in the ocean. Then water will flow out of the lake into the ocean, and the water level of the lake will decrease. That is:

$$x(a) > y(a) \Rightarrow \dot{x}(a) < 0$$

Similarly,

$$x(a) < y(a) \Rightarrow \dot{x}(a) > 0$$

As long as things are not too far from equilibrium, this behavior may be modeled *linearly*:

$$\dot{x} = k(y - x)$$

where $k > 0$. The parameter k is the *coupling constant*. If it is large, the water level in the lake responds strongly to differences with the water level in the ocean—the channel is wide. If it is small, the response is correspondingly small—the channel is narrow.

In principal the coupling constant need not be constant—it could depend upon time. (Maybe the channel is eroding and changing its form.) Normally for time periods of interest it is not unrealistic to suppose that k is constant.

This equation can be rewritten

$$\dot{x} + kx = ky$$

The tide in the ocean is approximately periodic. If we approximate the ocean level by a sinusoid, we might as well fix the clock so the sinusoid is $B\cos(\omega t)$. The differential equation is then

$$\dot{x} + kx = kB\cos(\omega t)\,.$$

The Mathlet illustrates this equation for input amplitude $B = 1$.

The period P of tidal action is approximately 12 hours. The relationship between period and circular frequency ω is

$$\omega = \frac{2\pi}{P}$$

so for tides ω is approximately $2\pi/12$ radians/hour or about 0.5 radians/hour.

The solution. The general solution to this linear equation is of the form

$$A\cos(\omega t - \phi) + ce^{-kt}$$

for constants A and ϕ which depend only on the equation, and c which is an arbitrary constant of integration. The second term is the *transient*, and dies off as t grows large. The first term is the *steady state solution*.

The amplitude of the system response, A, is a multiple of the input amplitude, and the ratio $g = A/B$ is the *gain*.

The constant ϕ is the *phase lag*. The tide in the bay trails the tide in the ocean, and ϕ expresses this in terms of radians. It can also be expressed in terms of the *time lag* t_0:

$$\cos(\omega t - \phi) = \cos(\omega(t - t_0))$$

where

$$t_0 = \phi/\omega$$

The Mathlet also illustrates a method of solution, i.e. of finding the constants A and ϕ. The method is to see the equation as the real part of a complex-valued equation with a complex exponential on the right hand side:

$$\dot{z} + kz = kBe^{i\omega t}$$

For any fixed complex number s (not just purely imaginary ones such as $i\omega$), the equation

$$\dot{z} + kz = kBe^{st}$$

has a unique solution which is a multiple of the exponential input signal,

$$z_p = W(s)e^{st}$$

where $W(s)$ is independent of t (though it does depend upon s). (An exception occurs if $s = -k$; this is "resonance." It does not concern us here because for us s will be purely imaginary.) By plugging this expression into the equation, you find

$$W(s) = \frac{k}{p(s)}$$

where $p(s) = s + k$ is the characteristic polynomial of the linear equation. (This is a special case of the "exponential response formula.") In particular,

$$W(i\omega) = \frac{k}{i\omega + k}$$

If you click on [Nyquist plot], a picture of part of the complex plane appears, with the trajectory of $W(i\omega)$ displayed.

Now, to recover a solution of the original equation, we take the real part of z_p. In order to recover A and ϕ, we begin by writing $W(i\omega)$ in polar form:

$$W(i\omega) = ge^{-i\phi}$$

Note that $\phi = -\mathrm{Arg}(W)$ and $g = |W|$.

Then

$$z_p = W(i\omega)Be^{i\omega t} = ge^{-i\phi}Be^{i\omega t} = gBe^{i(\omega t - \phi)}$$

Now the real part is easy to write down:

$$x_p = gB\cos(\omega t - \phi)$$

Thus: $g(\omega) = |W(i\omega)|$ is the *gain* and $\phi(\omega) = -\mathrm{Arg}(W(i\omega))$ is the *phase lag*. Clicking on [Bode plots] causes graphs of these two functions of ω to be drawn.

(These are not truly Bode plots. The amplitude Bode plot graphs the *log* of the amplitude against the *log* of the frequency. The phase lag is already in some sense a logarithm (it occurs as an exponent), and the true phase Bode plot graphs the phase lag against the log of the frequency.)

It's easy to compute the magnitude and argument of $W(i\omega)$ more explicitly: the gain is

$$g = \frac{k}{\sqrt{k^2 + \omega^2}}$$

For the phase lag, rationalize the denominator:

$$W(i\omega) = \frac{k}{k^2 + \omega^2}(k - i\omega)$$

This complex number lies in the lower right quadrant, so $\phi = -\mathrm{Arg}(W(i\omega))$ is the angle between 0 and $\pi/2$ such that $\tan(\phi) = \omega/k$, i.e.

$$\phi = \arctan(\omega/k)$$

Questions. 1. Notice from the Mathlet that when the system response curve (yellow) crosses the input signal curve (cyan), it appears to have horizontal tangent. Is this really so? Why?

2. The Nyquist plot (for $\omega \geq 0$) appears to be independent of the value of k, and to form a semicircle. Is this really true? Why? If so, what is the center and radius of the circle?

3. For any value of k, as ω gets large it appears that $A \to 0$. Is this right? Can you be more precise about how? That is, for ω very large, can you approximate A by a simpler expression that what we derived above?—maybe just a (negative) power of ω?

4. For any value of k, as ω gets large it appears that $\phi \to \pi/2$. Is this right?

5. How about the behavior of A and ϕ for small values of ω? Clearly $g(0) = 1$ and $\phi(0) = 0$, so linear approximation gives

$$g(\omega) \simeq 1 + a\omega \quad , \quad \phi(\omega) \simeq b\omega$$

for small ω. The Mathlet gives some indication of the values of a and b. What are they in fact? Does the Mathlet bear out your calculation?