Complex Exponential

Problem

(a) Find an expression for $\sin(4t)$ in terms of sums of powers of $\sin t$ and $\cos t$ by using $(e^{it})^4 = e^{4it}$ and Euler's formula.

The Mathlet `Complex Exponential` will probably be useful in understanding the rest of this problem. Open it and explore its functionalities. The `Help` button lists most of them. Notice that in the left window, the real part a ranges between -1 and 1, while the imaginary part b ranges from -8 to 8. You use the left-hand window to pick out a complex number $a + bi$. When you do, a portion of the line through it and zero is drawn. This line is parametrized by $(a + bi)t$. At the same time, the curve parametrized by the complex-valued function $e^{(a+bi)t}$ is drawn on the right window.

(b) Sketch the function $f(t) = e^{-t}\cos(2\pi t)$ for t between -1 and 1. Write down a value of $a + bi$ such that $f(t)$ is the real part of $e^{(a+bi)t}$; sketch a graph of $e^{(a+bi)t}$ for this value of $a + bi$; and sketch the curve in the complex plane parametrized by this complex-valued function.

For each of the following questions, explain your answer using Euler's formula $e^{(a+bi)t} = e^{at}(\cos(bt) + i\sin(bt))$.

(c) For what values of $a + bi$ is the curve $e^{(a+bi)t}$ a circle?

(d) For what values of $a + bi$ is the curve $e^{(a+bi)t}$ a ray? What rays are possible?

(e) For what values of $a + bi$ does the curve $e^{(a+bi)t}$ converge towards zero as t grows?

(f) For what values of $a + bi$ is the curve $e^{(a+bi)t}$ a spiral which moves away from the origin and curls counterclockwise as t increases?

(g) Drop the cursor at a nonzero complex number $a + bi$ and look at the trajejctory of $e^{(a+bi)t}$. (The right plane shows only the part of this curve which corresponds to values of $(a + bi)t$ which fit on the left plane.) For what other values of z is the trajectory of e^{zt} the same as the trajectory of $e^{(a+bi)t}$?