Damped Vibration

Problem 1

Saloon doors can swing through the door frame. A good door damper will slow a swinging door down so it does not swing through the door frame—unless you shove the door hard toward the frame. In that case, it will swing through and return from the other side. This is a characteristic of "overdamping."

This problem will study this effect, using the Mathlet **Damped Vibration**. Open the applet. It's about solutions of the second order homogeneous equation $\ddot{x} + b\dot{x} + kx = 0$. The initial conditions are set using the box at left. In it the horizontal direction gives $\dot{x}(0)$ and the vertical direction gives $x(0)$. The right graphing window displays the corresponding solution.

Move the cursor around in the initial conditions box and observe the behavior of the left end of the graph in the right window. Verify for yourself that the slope increases when the horizontal coordinate increases, and that the value increases when the vertical coordinate increases. These coordinates can also be controlled using the sliders along the edge of the initial conditions box.

As you decrease the damping constant b, you will see that the sytem becomes "under-damped": the solutions oscillate. In this problem, though, we will study the case in which $b = 2$ and $k = 0.75$. (The applet won't let you choose $k = 0.75$. Approximate it by 0.74.) Check that this is overdamped: the characteristic polynomial has two distinct real roots, and the solutions don't oscillate.

Now set $x(0) = 0.50$. Set the $\dot{x}(0)$ slider at -1.00 and start to increase it slowly. Watch the effect on the solution curve. At first it swings through $x = 0$, but soon seems not to. You can get a better picture by zooming in using one of the power-of-ten buttons. Work with this till you come up with the smallest value of $\dot{x}(0)$ which does not result in a solution crossing the x axis. (It will be a negative number.)

(a) State the value you discover.

(b) Now find the general solution of this ODE, with these values of k and b. Express the constants of integration in terms of $\dot{x}(0)$ (using $x(0) = 0.5$).

(c) Finally, verify that there is a number v such that the solution never takes on the value zero for $t > 0$ if $\dot{x}(0) \geq v$, while it does if $\dot{x}(0) < v$. Does v agree with your measurement on the Mathlet? What is the solution with $\dot{x}(0) = v$?

Problem 2

This problem is about solutions of the second order homogeneous equation $\ddot{x} + b\dot{x} + kx = 0$. The initial conditions are set using the box at left. In it the horizontal direction gives $\dot{x}(0)$ and the vertical direction gives $x(0)$. The right graphing window displays the corresponding solution.

Move the cursor around in that box and observe the behavior of the left end of the graph in the right window. Verify for yourself that the slope increases when the horizontal slider is moved right, and that the value increases when the vertical slider is moved up.

(a) Now set $b = 0.00$, $k = 2.50$. What is the differential equation we are looking at? Pick any value of the initial conditions you want (except $(0,0)$). State them, and measure the horizontal (time) coordinate of the first and second maxima of the solution. Their difference is the period, P. On the other hand, what is the computed period of the general (nonzero) solution of the ODE? Do they agree?

(b) Now let's introduce some damping: set $k = 2.50$ and $b = 0.50$ The solution is of the form $x = Ae^{-bt/2}\cos(\omega_d t - \phi)$, for some choice of amplitude A and phase lag ϕ which depend upon the initial conditions. $b = 0.50$ for us. The circular frequency ω_d is called the "damped frequency." What is ω_d? (You have to solve for it using the theory.) What then is the quasiperiod $P = 2\pi/\omega_d$? What effect has adding a dashpot had on the period of oscillation of this system?

Now, in the Mathlet, choose initial conditions as you like (except $(0,0)$). State them, and measure the coordinates of the first, second, and third maxima of the solution: $(t_1, x_1), (t_2, x_2), (t_3, x_3)$. You may have to zoom in or out using the powers of ten buttons to get an accurate view of where the maxima occur. Compute the differences $t_2 - t_1$ and $t_3 - t_2$, and the ratios x_2/x_1 and x_3/x_2. Do the time differences match your computation of P?

(c) From the formula, $x(t) = 0$ just where $\cos(\omega_d t - \phi) = 0$, and these are spaced out in intervals of $P/2$. This is independent of the value of b. But it's not so clear that the maxima of $x(t)$ are spaced the same way that the maxima of $\cos(\omega_d t - \phi)$ are. They certainly don't occur at the same time that the maxima of $\cos(\omega_d t - \phi)$ do if $b \neq 0$. How do I know that?

(d) But if your measurements were made accurately, you should find that the time interval between successive maxima seems to equal the quasi-period (and so independent of initial condition). Verify this observation by computation: that is, show that the difference between successive maxima of the function $e^{-bt/2}\cos(\omega_d t - \phi)$ is P. (Do this by setting the derivative to zero. You will want to say how much you have to increase t before a given value of $\tan(\omega_d t - \phi)$ re-occurs. Remember, maxima and minima alternate.)

(e) Now use what you learned in (d) to show that the ratio of one maximum to the prior one is independent of which maximum you chose, and also independent of the choice of initial conditions. How well does it match with what you measured in (b)?

Problem 3

An important use of critical damping is to bring a system to equilibrium in the shortest possible time. This problem illustrates with

$$x'' + bx' + kx = 0, x(0) = x_0, x'(0) = x_0'.$$

You can think of this as modeling a mass-spring-dashpot system, whose mass is 1, damping constant is b, and spring constant is k ; the number x_0 and x_0' are the initial position and velocity of the mass.

(a) Using the value $b = 1.2 + .04N$, where N is your recitation number, find the value k for the spring constant which gives critical damping.

(b) In the figlet, you set the values of b and k with two sliders; you set the initial conditions x_0 and x_0' by choosing a point in the splane on the left: either directly by

using the mouse. or by using the vertical and horizontal sliders. You can also magnify the x-axis, and get the coordinates of any point on the graph by mocing the mouse in the region just below the line $x = 0$. Play around with the sliders and mouse until you see how it all works.

Then set $x_0 = 1$ and $x_0' = 0$; the spring is stretched so that the mass is a 1 and at rest; at time $t = 0$ it is released and the resulting graph shows its subsequent motion.

To get some feel for how accurate the screen display is:

Set b to the value in part **(a)** , set the scale to max $x = 1$ to see the overall picture; then adjust k to what seems visually to be the critically damped solution curve.

Then go to the scale where max$x = .01$, and readjust k. Record this experimental value of k alongside your theoretical value in part **(a)**. Then set k to the theoretical value, and determine how many seconds (to two decimal places) it takes x to fall to .001.

Overdamping: Keeping k fixed, increase the damping b by four successive increments of .10, in each case recording how many seconds it takes x to fall to .001. Make a table showing the five values of b and the corresponding times.

Underdamping: Decrease the damping to $b-.1$ and $b-.2$; now the mass overshoots, but record how many seconds it takes to get and stay within .001 if x $= 0$.

From your table, if one wants to adjust the damping so the masss returns as quickly as possible to be permanently near the equilibrium position $x = 0$, is it better to err on the side of overdamping or underdamping?

(c) Set $x_0 = .5$ and use the scale max$x = 1$, experiment with negative values of x_0'. Evidently, if x_0' is sufficiently negative, the graph will cross the x-axis: i.e., even though the system is critically damped, the mass will overshoot the equilibrium position $x = 0$, because it is being given an initial push in the negative direction.

The idea is to determine both experimentally and then theoretically how hard this negative initial push must be, in other words, find the value of K such that

$$x_0' < -K \text{ if and only if the mass crosses the equilibrium position.}$$

i) Use max$x = .01$. Record the value of K to you decimal places.

ii) Away from the terminal, find K theoretically, by solving in general the equation for the critically damped ODE, writing for convenience the spring constant as $k = a^2$, with b expressed in terms of a. (Your solution will involve x_0, x_0', a.)

Assume $x_0 > 0$ and use your general solution to determine the value of K, in terms of the other parameters. (Hint: we want to know whether the mass will cross the equilibrium point for some positive value of t)

Compare this value with the experimental value for K you found in (i), using the values of x_0 and b you used there.

Problem 4

(a) Take the damped sinusoidal function $x = Ae^{-at}cos(wt - \phi)$ with $A > 0$. By definition, its quasi-period is the period of the corresponding undamped sinusoid,

$P = 2\pi/\omega$. This is the time between successive maxima of the undamped cosine function. Show that the time between successive maxima of the damped sinusoid is also $P = 2\pi/\omega$. If $a > 0$, the maxima of x do not occur when its graph touches the "envelop" e^{-at}. When a is small, the maxima occur either just before or just after the tangecy. Which, and why?

(b) Suppose one maximum occurs at $t = t_1$ and the next at $t = t_2$, and we measure that $t_2 - t_1 = 4$. What is ω? Next, we discover that the ratio $x(t_2)/x(t_1) = 1/2$. This lets us determine a; what is it?

(c) The damped sinusoid we have been studying is a solution to the equation $x'' + bx' + kx = 0$ for suitable values of the damping constant b and the spring constant k. What are b and k, both exactly and as decimals rounded to the third decimal place.

(d) Finally, invoke the Mathlet Damped Vibrations and set the parameters to the values of b and k you found in (c). Measure the time and the height of the first and second ,maxima of the solutions corresponding to the initial values $(x'(0), x(0)) = (1, 0)$ and $(x'(0), x(0)) = (1, 1)$ and verify that these measurements agree with the values we started with in (b). (Note that in the left window, the horizontal axis measures x' and the vertical measures x. Also, the readout from the graphing window rollover gives the selected value of t and the value of $x(t)$. Also, you may have to use the zoom buttons to find a maximum if it's off the screen) This should reassure you that the time between maxima and the ratio of neighboring maxima is independent of which solution you are working on. They are also independent of which maxima you choose, and they give a practical way to determining the system parameters from an unforced solution. In practice you would do better to compare two maxima which differ by more than just on period; you'd improve the error bar that way.

Problem 5

In this problem the differential equation is $x'' + 3x' + 2x = 0$ for $x = x(t)$, with initial condition $x(0) = 1/4$, $x'(0) = v$.

(a) Solve for $x = x(t)$.

(b) Find the range of values of v for which the solution $y(t)$ will pass through $x = 0$ for some positive value of t. (Hint: set $x = 0$ and express e^t in terms v. Also, you will find that $v > 0$ doesn't work, but keep going).

(c) Open the applet "Damped Vibration" and see how it works . Then use it to check your answer to part (b), and record how close you could get using the applet to the answer predicted by theory. Notes: If this differential equation is viewed as modeling an overdamped spring-mass system, then you'll see why v must be sufficiently large negative value: if you displace the mass initially to $x_0 = 1$, then you 've got to give the mass an initial kick in wihch direction to get it to overshoot the equilibrium position $x = 0$? For v in the range for which the mass does cross the equilibrium position, how many times does it do so?