Isocline

Problem 1

In this problem you will study solutions of the differential equation

$$\frac{dy}{dx} = y^2 - x^2\,.$$

Solutions of this equation do not admit expressions in terms of the standard functions of calculus, but we can study them anyway using the direction field.

(a) Draw a large pair of axes and mark off units from -4 to $+4$ on both. Sketch the direction field given by our equation. Do this by first sketching the isoclines for slopes $m = -2$, $m = 0$, and $m = 2$. On this same graph, sketch a couple of solutions.

Having done this, we will continue to investigate this equation using one of the Mathlets. So invoke `http://www-math.mit.edu/daimp` in a web browser and select `Isoclines` from the menu. (If you are using a Mac, you may find it difficult to use Firefox for these applets and should probably pick a different browser.) Play around with this applet for a little while. The Mathlets have many features in common, and once you get used to one it will be quicker to learn how to operate the next one. Clicking on "Help" pops up a window with a brief description of the applet's functionalities.

Select $y' = y^2 - x^2$ from the pull-down menu, and verify your work in **(i)**. (Nothing to turn in here.)

By clicking on the graphing window, cause several solutions to be drawn. It appears that solutions fall into two main types, according to their behavior as x becomes large: some increase without bound, while others decrease without bound.

(b) It appears that there is some constant y_0 such that if y is a solution with $y(0) > y_0$ then y becomes large as x becomes large, while if $y(0) < y_0$ then y decreases as x increases. By experimenting with the applet, find y_0 to within 0.01: that is, find two numbers which differ by 0.02 and have y_0 between them. Explain briefly what you did.

(c) Is there some function $f(x)$ such that $y(x) > f(x)$ for all $x > 0$ whenever y is a solution with $0 < y(0) < y_0$ (i.e., which is falling for sufficiently large x)? You can use the Mathlet to get some ideas, but then you should find a way to explain the answer you come up with.

(d) Suppose that a solution y has a maximum at the point (a, b). What can you say about the relationship between a and b? Explain.

Problem 2

We will study solutions of the following differential equation

$$\frac{dy}{dx} = y^2 - x.$$

(a) The first part you have to do by hand. For the above ODE, using the scale 1 unit = 1/2 inch and the interval [-4,4] on both axes, draw in dotted lines a sketch of the isoclines corresponding to the slopes 0, 1, -1, 2, -2, with the accompanying line elements. Then draw in five solution curves, which illustrate the general range of behavior.

(b) At the computer, first get familiar with the operation of "Isoclines" by using it to check your work in part (a). Then answer the following questions.

i) From the screen, it looks like there is a critical initial value K which determines the long-term behavior of the solutions:

$y(0) < K \Rightarrow$ the solution $y(x)$ rises, then falls to 0 for some positive x

$y(0) > K \Rightarrow$ the solution $y(x)$ increases without bound.

Determine the value of K to within .02. Say briefly what you did.

ii) Let $y(x)$ be such that $y(0) < K$, i.e. rise-and-fall solution. Estimate $y(100)$, with error bounds, and a brief reason.

iii) Are all the solutios in (ii) bounded for $x \geq 0$ by some number M- i.e, is there a number M such that $y(x) \leq M$, for all $x \geq 0$ and for all such rising and falling solutions $y(x)$? Or is there no such number M? Give a brief reason.

Problem 3

(a) Select the differential equation $y' = x - 2y$. Click up some solution curves, and then watch the isocline move as you drag the m slider. The isoclines are straight lines, and it appears that one of them is a solution. Check this analytically: which straight line is a solution to $y' = x - 2y$?

For a general differential equation $y' = F(x, y)$, if an isocline is also a solution is it necessarily a straight line? Explain.

(b) Invoke the differential equation $y' = y^2 - x$. Click up some solutions, enough so that the separatrix becomes apparent. A separatrix is a solution curve such that solutions above it behave entirely differently as t grows large than do solutions below it. In this instance, there is just one separatrix; write $f(x)$ for this solution. What is the value $f(4)$, approximately? What is the largets m for which the isocline $y^2 - x = m$ lies entirely below the separatrix in the time range shown in the window? Explain why $y^2 - x = 0$ lies entirely below the separatrix. Is there any $m > 0$ for which the parabola $y^2 - x = m$ lies below the separatrix for all large time?

(c) Where do the maxima of solutions of $y' = y^2 - x$ occur?

(d) Where do points of inflection occur? For this, take $y' = y^2 - x$, differentiate it, and substitute $y^2 - x$ for y' in the result. Submit a sketch of this curve, and satisfy yourself that solutions do have points of inflection along that curve.