Poles and Vibrations

Problem 1

Definition Let $P(s)$ and $Q(s)$ be polynomials with no common factors. Then $R(s) = P(s)/Q(s)$ is called a rational function. A **pole** of $R(s)$ is a complex zero of $Q(s)$, i.e., a point where $|R(s)|$ becomes infinite.

(a) We know $e^{at}cosbt = Re(e^{(a\pm bi)t})$ and $e^{at}sinbt = \pm Im(e^{(a\pm bi)t})$. The complex numbers $a \pm bi$ are also connected to these functions by the fact that they are poles of $\mathcal{L}(e^{at}cosbt)$ and $\mathcal{L}(e^{at}sinbt)$. Show this.

(b) Start the applet "Poles And Vibrations". As usual, try it out for a little while: First practice with just one of the two functions by setting $B = 0$. Change one of A, a, b, leaving the other two fixed, and study the effect on the graph. Note the plot in the lower right of the screen represents the poles of $\mathcal{L}(f)$, i.e. $a \pm bi$ and $c \pm di$. Now, fix A and b at their maximum values, set $a = 0.06$ and report how many minima $f(t)$ has before it leaves the screen. Also, report the approximate location (in the form $t = k\pi$) on the last on-screen minimum.

(c) Now look at the effect of summing two sinusoidal oscillations: Set $a = c = 0$ and $a = b = 1$.

(i) Frequencies far apart: Set $d = 8$ and $b = 1$ and draw a rough sketch of the screen.

(ii) Frequencies close together: Set $d = 7$ and $b = 6.5$, and report the approximate circular frequency of the beats by estimating their period from the screen.

(d) Definition: For a function $f(t) = e^{at}$, or $e^{at}cosbt$, or $t^n e^{at}$, we call a the **exponential growth rate**. If $a > 0$ then $f(t) \to \infty$ and if $a < 0$ then $f(t) \to 0$. For the sum of two or more functions the exponential growth rate is the largest growth rate among the summands.

Set $A = B = 1$, so $f(t)$ is the sum of two oscillations. Use the mouse to drag the poles around the complex plane, observing what happens to the graph of the resulting function.

(i) Where must the poles of $\mathcal{L}f(s)$ be if $f(t) \to 0$ as $t \to \infty$?

(ii) In terms of a and c, what is the exponential growth rate of $f(t)$?

(iii) What is the yellow function $g(t)$ telling you? What makes it shift between its two possible forms as you move the poles around?

(iv) The Laplace transforms of two functions $h_1(t)$ and $h_2(t)$ are given below. For each, tell what the exponential growth rate is without actually calculating either function.

$$H_1 s) = \frac{s^2 - 3s + 2}{(s^2 - 6)(s^2 + 2s + 5)}, \quad H_2(s) = \frac{s^2 - 3s + 2}{(s^2 - 4)(s^2 + 2s + 5)}$$