Beats

What beats are. Musicians tune their instruments using "beats." Beats occur when two very nearby pitches are sounded simultaneously. The Mathlet Beats illustrates this, in the simplest case when both periodic functions are sinusoidal. It allows one to study how beats materialize as two frequencies converge, and how beats respond to changes in relative phase and amplitude of the two sinuoids.

In the applet, one sinusoid is taken as the "reference wave," with fixed frequency, amplitude, and phase:

$$f(t) = \sin(t)$$

The other sinuoid is variable in all three dimensions:

$$g(t) = A\sin(\omega t - \phi)$$

One case admits a simple discussion, namely when the two amplitudes are equal and the relative phase is zero: $A = 1$, $\phi = 0$. Then the trig identity

$$\sin((\alpha+\beta)t) + \sin((\alpha-\beta)t) = 2\cos(\beta t)\sin(\alpha t) \qquad (1)$$

with

$$\alpha = \frac{1+\omega}{2} \quad , \quad \beta = \frac{1-\omega}{2}$$

gives us the equation

$$f(t) + g(t) = \sin(t) + \sin(\omega t) = 2\cos\left(\left(\frac{1-\omega}{2}\right)t\right)\sin\left(\left(\frac{1+\omega}{2}\right)t\right). \qquad (2)$$

The constant α determines the "beat pitch"; it is the average of the two constituent frequencies, and it is the pitch we hear. The small constant $|\beta|$ is the "beat (circular) frequency." You should think of the first factor as modulating the amplitude of the sinusoid given by the second factor.

The Mathlet has an option of displaying the graph of $\sin(\alpha t)$. When ϕ is varied away from zero, the beat pitch is still close to α, though the relationship between $f(t)+g(t)$ and $2\cos(\beta t)\sin(\alpha t)$ becomes more complex. But when A is varied away from 1 this is no longer a good approximation of the pitch. We know this for sure because in the limiting case, when $A = 0$, we obtain a sinusoid with circular frequency 1, independent of ω.

Here is a general equation for an "envelope" for the beat:

$$e(t) = \sqrt{1 + A^2 + 2A\cos((1-\omega)t)}, \qquad (3)$$

The Mathlet provides evidence for the claim that when $\phi = 0$,

$$|f(t) + g(t)| \leq e(t)$$

This is explored further in the Questions below.

What beats are not. Many differential equations textbooks present beats as a system response when a harmonic oscillator is driven by a signal whose frequency is close to the natural frequency of the oscillator. This is true as a piece of mathematics, but it is almost never the way beats occur in nature. The reason is that if there is any damping in the system, the "beats" die out very quickly to a steady sinusoidal solution, and it is that solution which is observed.

Explicitly, the "Exponential Response Formula" shows that the equation

$$\ddot{x} + \omega_n^2 x = \cos(\omega t)$$

has the periodic solution

$$x_p = \frac{\cos(\omega t)}{\omega^2 - \omega_n^2}$$

unless $\omega = \omega_n$. If ω and ω_n are close, the amplitude of the periodic solution is large; this is "near resonance." Adding a little damping won't change that solution very much, but it will convert homogeneous solutions from sinusoids to *damped* sinusoids, i.e. transients, and rather quickly any solution becomes indistinguishable from x_p.

So beats do not occur this way in engineering situations. But they do occur. They are used for example in reconstructing an amplitude-modulated signal from a frequency-modulated ("FM") radio signal. The radio receiver produces a signal at a fixed frequency ν, and adds it to the received signal, whose frequency differs slightly from ν. The result is a beat, and the beat frequency is the audible frequency.

Differential equations textbooks also always arrange initial conditions in a very artificial way, so that the solution is a sum of the periodic solution x_p and a homogeneous solution x_h having exactly the same amplitude as x_p. They do this by imposing the initial condition $x(0) = \dot{x}(0) = 0$. This artifice puts them into the simple situation $a = b$ mentioned above. For the general case one has to proceed as we did, using complex exponentials.

Questions. 1. Prove the trig identity (1) using complex numbers.

2. Verify that (3) reduces to (2) when $A = 1$; that is,

$$e(t) = 2|\cos(\beta t)|\,.$$

3. By writing both sinusoids (still with $\phi = 0$) as imaginary parts of complex exponentials, explain where the beat envelope $g(t)$ came from.

4. Now address the phase issue. Using the Mathlet, suggest a relationship between the phase of the beat and the phase of $g(t)$ (relative to the reference sinusoid $f(t) = \sin t$). Then verify this prediction by providing a variant of the formula for $e(t)$ taking ϕ into account.