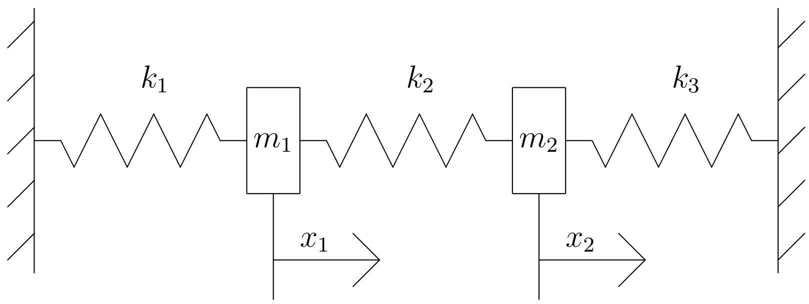
Coupled Oscillators

We have two masses and three springs, arranged as in the picture. What are the possible motions of this system, even in the absence of forcing (and damping!)?



First set up parameters: The two masses are m_1 and m_2; the three spring constants are k_1, k_2, and k_3.

This system has a "rest" position, when the forces on the two masses balance and the velocities are zero. Set up the two displacement parameters x_1 and x_2 so that both are zero in that situation, and each becomes positive when the corresponding mass moves to the right. So the middle spring is relaxed whenever $x_1 = x_2$.

Each of the masses has two forces acting on it, and Newton's law says:

$$\begin{aligned} m_1\ddot{x}_1 &= -k_1 x_1 + k_2(x_2 - x_1) \\ m_2\ddot{x}_2 &= -k_3 x_2 + k_2(x_1 - x_2) \end{aligned}$$

or

$$\begin{aligned} m_1\ddot{x}_1 &= -(k_1 + k_2)x_1 + k_2 x_2 \\ m_2\ddot{x}_2 &= k_2 x_1 - (k_2 + k_3)x_2 \,. \end{aligned}$$

We have is a system of *second order* equations, but we can still represent it using matrices. Let's write $\mathbf{x}$ for the column vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, so

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \ddot{\mathbf{x}} = \begin{bmatrix} -(k_1 + k_2) & k_2 \\ k_2 & -(k_2 + k_3) \end{bmatrix} \mathbf{x} \,.$$

As usual, we can divide through by the masses, and obtain an equation with $\ddot{\mathbf{x}}$ alone on the left. Write

$$B = \begin{bmatrix} -(k_1+k_2)/m_1 & k_2/m_1 \\ k_2/m_2 & -(k_2+k_3)/m_2 \end{bmatrix}$$

so that our equation is just

$$\ddot{\mathbf{x}} = B\mathbf{x}\,.$$

To analyze this we will need to convert it to a larger system of first order equations, using the companion matrix idea. So we introduce a new vector variable $\mathbf{y}$ and declare that

$$\dot{\mathbf{x}} = \mathbf{y}\,.$$

Then our equation looks like this:

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \dot{\mathbf{y}} \end{bmatrix} = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

where I is the 2×2 identity matrix.

This is a homogeneous linear first order system, and we can analyze it using eigenvectors and eigenvalues. This is a 4×4 matrix, but of a pretty simple form. The eigenvector equation is

$$\begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$$

or

$$\mathbf{y} = \lambda\mathbf{x}\,, \quad B\mathbf{x} = \lambda\mathbf{y}\,.$$

Substituting the first equation into the second we find

$$B\mathbf{x} = \lambda^2\mathbf{x}\,,$$

which is to say that $\mathbf{x}$ is an eigenvector for the 2×2 matrix B with eigenvalue λ^2.

We have learned that the four eigenvalues of A are the square roots of the two eigenvalues of B. And the eigenvectors are gotten by putting $\lambda\mathbf{x}$ below the $\mathbf{x}$.

Let's do an example: suppose that the spring constants are equal—say all with value k—and the two masses are equal—say with value m. Then

$$B = \omega^2 \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \quad \omega = \sqrt{\frac{k}{m}}\,.$$

The characteristic polynomial of B is

$$p_B(\lambda) = \lambda^2 + 4\omega^2\lambda + 3\omega^4 = (\lambda + \omega^2)(\lambda + 3\omega^2)$$

so its eigenvalues are $-\omega^2$ and $-3\omega^2$.

That says the eigenvalues of A are $\pm i\omega$ and $\pm\sqrt{3}i\omega$.

We're almost there. The eigenvectors for B: For $-\omega^2$ we want to find a nonzero vector killed by $B - (-\omega^2 I) = \omega^2 \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$; $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ will do. The matrix is symmetric so eigenvectors for different eigenvalues are orthogonal; an eigenvector for value $-3\omega^2$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

So the eigenvectors for A are given by

$$\lambda = \pm i\omega : \begin{bmatrix} 1 \\ 1 \\ \pm i\omega \\ \pm i\omega \end{bmatrix} \quad , \quad \lambda = \pm\sqrt{3}i\omega : \begin{bmatrix} 1 \\ -1 \\ \pm\sqrt{3}i\omega \\ \mp\sqrt{3}i\omega \end{bmatrix}$$

This gives us exponential solutions!

$$e^{i\omega t}\begin{bmatrix} 1 \\ 1 \\ i\omega \\ i\omega \end{bmatrix} \quad , \quad e^{i\sqrt{3}\omega t}\begin{bmatrix} 1 \\ -1 \\ \sqrt{3}i\omega \\ -\sqrt{3}i\omega \end{bmatrix}$$

and their complex conjugates. We can get real solutions by taking real and imaginary parts. Let's just write down the top halves; the bottom halves are just the derivatives.

$$\begin{bmatrix} \cos(\omega t) \\ \cos(\omega t) \end{bmatrix} \quad , \quad \begin{bmatrix} \sin(\omega t) \\ \sin(\omega t) \end{bmatrix} \quad , \quad \begin{bmatrix} \cos(\sqrt{3}\omega t) \\ -\cos(\sqrt{3}\omega t) \end{bmatrix} \quad , \quad \begin{bmatrix} \sin(\sqrt{3}\omega t) \\ -\sin(\sqrt{3}\omega t) \end{bmatrix}$$

The first two combine to give the general sinusoid of angular frequency ω for x_1, and $x_2 = x_1$. In this mode the masses are moving together; the spring between them is relaxed.

The second two combine to give a general sinusoid of angular frequency $\sqrt{3}\omega$ for x_1, and $x_2 = -x_1$. In this mode the masses are moving back and forth relative to each other.

These are "normal modes." From Wikipedia:

> A normal mode of an oscillating system is a pattern of motion in which all parts of the system move sinusoidally with the same frequency and with a fixed phase relation.

Behind the chaotic movement there are two very regular, sinuoidal motions. They happen at different frequencies, and that makes the linear combinations look chaotic. In fact, in this example, the two frequencies never match up, because $\sqrt{3}$ is an irrational number. Except for the normal mode solutions, no solutions are periodic.