Eigenvalue Stability Lab

Today we will use an interactive mathlet to understand the concept of eigenvalue stability. Special thanks to Professor Haynes Miller and Jean-Michel Claus for their tireless efforts in putting this mathlet together.

NAME____

1. **Open the Eigenvalue Stability Mathlet** Open the applet by navigating to

   `http://math.mit.edu/~jmc/daimp/EigenvalueStability.html`

   in your favorite web browser. If nothing comes up, you may have to install/enable the Java plugin and relaunch your browser. It should only take a minute.

2. **Stability Boundary**

   (a) Select one of the available numerical integration schemes from the pull-down menu in the lower left, and check the `Formula` box below. This formula displays the amplification factor g as a function of $z = \lambda \Delta t$.

   (b) Choose two (2) values of θ (θ_1 and θ_2) in the range $[0, 2\pi)$. These angles define two (2) points on the unit circle in the g-plane, $g = e^{i\theta_1}$ and $g = e^{i\theta_2}$. In the space below, calculate the corresponding values of z (z_1 and z_2).

   $\theta_1 =$ ____________________ $\theta_2 =$ ____________________

   $z_1 =$ ____________________ $z_2 =$ ____________________

(c) Now grab the θ slider in the upper right and, in turn, set it to each value of θ you selected above. As you change θ you see the yellow diamond in the main figure trace out the corresponding values of z (i.e., the stability boundary). Check your solution above by rolling over the yellow diamond in the main figure to see its complex coordinates.

(d) Select a different numerical integration scheme. Drag the θ slider from $\theta = 0$ to $\theta = 2\pi$ and observe how the stability boundary is traced out in the z-plane in real time.

3. **Eigenvalue Stability**

(a) Select either `Forward Euler (RK1)` or `RK2` from the pull-down menu.

(b) Consider the numerical solution of the following parameterized initial value problem:

$$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -a & 2a \\ -a & -3a \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad a \neq 0, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad t \in [0, 10].$$

Find the eigenvalues as a function of the parameter a.

$\lambda_1 =$ ____________________ $\lambda_2 =$ ____________________

(c) Let $a = 2\sqrt{5}$. For the method you selected in part (a), use the applet to determine the maximum allowable time step Δt which preserves eigenvalue stability.

$\Delta t =$ ____________________

(d) Computational resource restrictions and sufficient accuracy requirements demand that you use the time step $\Delta t = 0.1$. Find the range of the parameter a for which your numerical integration will be stable.

$a \in$ [____________, ____________]

4. **Pick A Scheme**

For each of the following scenarios, suggest a numerical integration scheme from the set

`{Forward Euler (RK1); Midpoint; Backwards Euler; Trapezoidal; RK2}`

and select the time step Δt. Provide a brief justification for your selection.

(a) The governing equation is $u_t = f(u) = -u^3$ with initial condition $u(0) = 1$. We need second-order accuracy, but evaluations of $f(u)$ are very expensive.

Method:________________________ $\Delta t =$ ____________

(b) There are competing populations of goldfish $G(t)$ and sharks $S(t)$. We have $dG/dt = G(1 - S)$ and $dS/dt = (G - 1)S$ with $G(0) = 1$ and $S(0) = 1$. We are interested only in the qualitative behavior of the solution (i.e. accuracy is not paramount).

Method:________________________ $\Delta t =$ ____________

(c) The governing equation is for a damped oscillator, $\theta_{tt} + \theta_t + \theta = 0$ with initial condition $\theta(0) = \pi/4$. Matrix inverses are disallowed, even in componentwise form. The output of interest is $\theta(1)$. We can only afford 50 matrix-vector multiplies.

Method:________________________ $\Delta t =$ ____________